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Isotopic Liftings of Clifford Algebras and Applications in Elementary Particle Mass Matrices

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Abstract Isotopic liftings of algebraic structures are investigated in the context of Clifford algebras, where it is defined a new product involving an arbitrary, but fixed, element of the Clifford algebra. This element acts as the unit with respect to the introduced product, and is called *isounit*. We construct isotopies in both associative and non-associative arbitrary algebras, and examples of these constructions are exhibited using Clifford algebras, which although associative, can generate the octonionic, non-associative, algebra. The whole formalism is developed in a Clifford algebraic arena, giving also the necessary pre-requisites to introduce isotopies of the exterior algebra. The flavor hadronic symmetry of the six u, d, s, c, b, t quarks is shown to be *exact*, when the generators of the *isotopic* Lie algebra $\mathfrak{su}(6)$ are constructed, and the unit of the isotopic Clifford algebra is shown to be a function of the six quark masses. The limits constraining the parameters, that are entries of the representation of the isounit in the isotopic group SU(6), are based on the most recent limits imposed on quark masses.

1 Introduction

Some limitations concerning the description of physical theories, owning non-canonical, non-unitary and non-Lagrangian character, have motivated investigations about a wider class

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of formalisms used to describe such theories, the so-called isotopies of mathematical structures. The isotopic lifting of such structures allows the physical theories to be described in a straightforward canonical, unitary and Lagrangian formalism [2–13], by maps from Lagrangian, linear and local theories to more general ones, involving a non-linear, non-local and non-Lagrangian character. These later are led to the former when formulated in an isospace, endowed with a new product in the context of the Clifford algebras, with respect to which the unit is now a fixed, but arbitrary, element ζ of the Clifford algebra. The inverse of such element is called *isotopic element*, and shall be used to define the new product that endows the *Clifford isotopic algebra*, to be precisely defined in this article. These isotopic concepts are entirely related to the q-deformations of algebraic structures, to which have a one-to-one correspondence to the isotopic liftings of algebras [12]. Although in e.g. [4] isotopies of symplectic and other geometries are included, the present paper presents for the first time the isotopies of Clifford algebras with significant applications.

In what follows we define isotopic Clifford algebras, and subsequently the formalism developed is applied in some aspects of Quantum Field Theory, e.g., the description of the flavor SU(6) symmetry as an *exact* symmetry among the six quarks, if they are to be viewed as components of an element of the carrier representation space of the isotopic group $\mathrm{SU}(6)_\zeta$ associated with the group SU(6), in the context of the isotopic Clifford algebra $\mathcal{C}\ell_{12,0}$. As a consequence, all six quarks must have the same mass in *isospace*, which brings an immediate constraint among the elements that constitute the matrix representing the Santilli's isounit, here emulated in a Clifford algebraic context. The isounit is shown to be a function of quark masses, whose original values are retrieved when an eigenvalue *iso*equation, or equivalently, the expected value defined in isospace, is used. The isotopic Lie algebra $\mathfrak{su}(6)_\zeta$, associated with the Lie algebra $\mathfrak{su}(6)$, is constructed in the context of the isotopic lifting of the Clifford algebra $\mathcal{C}\ell_{12,0}$. More generally, the isotopic lifting of $\mathfrak{su}(n)$ is described in the context of the isotopic lifting of the Clifford algebra $\mathcal{C}\ell_{2n,0}$, emulating a similar construction in [33].

We illustrate the general method to be used, by firstly describing the flavor symmetry among the u, d and s quarks as an exact symmetry of the isotopic $\mathrm{SU}(3)_\zeta$ group, constructed via the isotopic lifting of the Dirac–Clifford algebra $\mathcal{C}\ell_{1,3}(\mathbb{C}) = \mathbb{C}\otimes\mathcal{C}\ell_{1,3}$. In this context the isotopic group $\mathrm{SU}(3)^f_\zeta\times \mathrm{SU}(2)_\zeta\times \mathrm{U}(1)_\zeta$ is obtained using solely the isotopic lifting of $\mathcal{C}\ell_{1,3}(\mathbb{C})$. Here $\mathrm{SU}(3)^f$ denotes the flavor group SU(3) and has nothing to do with the SU(3) gauge group associated with strong interactions. Hereon we omit the index ^{f} and denote $\mathrm{SU}(3)^f$ solely by SU(3). We emphasize that the isotopies of SU(3) and the proof of their local isomorphism to the conventional SU(3) symmetry were first proved in [13] and papers quoted therein. After introducing the iso-Gell-Mann matrices, as particular cases of the most general representation in the isotopic $\mathfrak{su}(3)$ Lie algebra, analogously to [18–20], the behavior of some quantum operators acting on the carrier fundamental representation space of the isotopic SU(3) group is investigated.

In terms of its applications, the main aim of this paper is to obtain an exact flavor symmetry encompassing all the six quarks via the isotopic lifting of the generators of the group SU(6). The parameters that define the isotopy are shown to be functions of the quark masses, and are delimited by the most recent limits of quark masses.

This article is organized as follows: in Sect. 2 a brief review on Clifford algebras is presented, and after discussing associative isotopies in Sect. 3, in Sect. 4 the isotopic liftings of non-associative algebras is presented. In Sect. 5 ζ-fields are presented, and in Sect. 6 we investigate the so-called Clifford admissible products in the context of the ζ-applications. In Sect. 7 the isotopic lifting of exterior algebras is introduced via Clifford isotopic algebras and in Sect. 8 and Sect. 9 a complete formulation concerning the isotopic lifting of spacetime algebra is presented in order to introduce the heterodimensional isotopic lifting of the group

SU(3). In Sect. 10 the more general case describing the isotopic generators of the Lie group SU(n) is constructed, in the light of the corresponding standard construction [33]. Finally in Sect. 11, applications to QFT are presented and we show how to suitably construct an isotopy in such a way that in isospace the six quarks have equal masses, and consequently the SU(6) flavor symmetry becomes an exact symmetry in isospace. In the Appendix the isotopic lifting of SU(6) is presented via the isotopic lifting of the Clifford algebra $\mathcal{C}\ell_{12,0}$.

2 Preliminaries

Let V be a finite n-dimensional real vector space and V^* denotes its dual. We consider the tensor algebra $\bigoplus_{i=0}^{\infty} T^i(V)$ from which we restrict our attention to the space $\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$ of multivectors over V. $\Lambda^k(V)$ denotes the space of the antisymmetric k-tensors, isomorphic to the k-forms vector space. Given $\psi \in \Lambda(V)$, $\tilde{\psi}$ denotes the *reversion*, an algebra antiautomorphism given by $\tilde{\psi} = (-1)^{[k/2]}\psi$ ($[k]$ denotes the integer part of k). $\hat{\psi}$ denotes the *main automorphism or graded involution*, given by $\hat{\psi} = (-1)^k\psi$. The *conjugation* is defined as the reversion followed by the main automorphism. If V is endowed with a non-degenerate, symmetric, bilinear map $g : V^* \times V^* \to \mathbb{R}$, it is possible to extend g to $\Lambda(V)$. Given $\psi = \mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^k$ and $\phi = \mathbf{v}^1 \wedge \cdots \wedge \mathbf{v}^l$, for $\mathbf{u}^i, \mathbf{v}^j \in V^*$, one defines $g(\psi, \phi) = \det(g(\mathbf{u}^i, \mathbf{v}^j))$ if $k = l$ and $g(\psi, \phi) = 0$ if $k \neq l$. The projection of a multivector $\psi = \psi_0 + \psi_1 + \cdots + \psi_n$, $\psi_k \in \Lambda^k(V)$, on its p-vector part is given by $\langle\psi\rangle_p = \psi_p$. Given $\psi, \phi, \xi \in \Lambda(V)$, the *left contraction* is defined implicitly by $g(\psi \lrcorner \phi, \xi) = g(\phi, \tilde{\psi} \wedge \xi)$. For $a \in \mathbb{R}$, it follows that $\mathbf{v} \lrcorner a = 0$. Given $\mathbf{v} \in V$, the Leibniz rule $\mathbf{v} \lrcorner (\psi \wedge \phi) = (\mathbf{v} \lrcorner \psi) \wedge \phi + \hat{\psi} \wedge (\mathbf{v} \lrcorner \phi)$ holds. The *right contraction* is analogously defined $g(\psi \llcorner \phi, \xi) = g(\phi, \psi \wedge \tilde{\xi})$ and its associated Leibniz rule $(\psi \wedge \phi) \llcorner \mathbf{v} = \psi \wedge (\phi \llcorner \mathbf{v}) + (\psi \llcorner \mathbf{v}) \wedge \hat{\phi}$ holds. Both contractions are related by $\mathbf{v} \lrcorner \psi = -\hat{\psi} \llcorner \mathbf{v}$. The Clifford product between $\mathbf{w} \in V$ and $\psi \in \Lambda(V)$ is given by $\mathbf{w}\psi = \mathbf{w} \wedge \psi + \mathbf{w} \lrcorner \psi$. The Grassmann algebra $(\Lambda(V), g)$ endowed with the Clifford product is denoted by $\mathcal{C}\ell(V, g)$ or $\mathcal{C}\ell_{p,q}$, the Clifford algebra associated with $V \simeq \mathbb{R}^{p,q}$, $p + q = n$. In what follows $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ denote respectively the real, complex and quaternionic fields.

3 Associative Isotopic Algebras

Consider a $\mathbb{C}$-associative algebra $\mathcal{A}$ endowed with a product AB denoted by juxtaposition, where $A, B \in \mathcal{A}$, and let $\zeta \in \mathcal{A}$ be a fixed, but arbitrary element of $\mathcal{A}$. The product $\diamond : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is given by

$$A \diamond B := A\zeta^{-1}B = (A\zeta^{-1})B = A(\zeta^{-1}B). \tag{1}$$

Clearly ζ is the unit of $\mathcal{A}$ with respect to the $\diamond$-product, since $A \diamond \zeta = \zeta \diamond A = A$, for all $A \in \mathcal{A}$. Since ζ is assumed to be always invertible, the product $\diamond$ is not automorphic to the product of the original algebras [22].

Given $A \in \mathcal{A}$, ζ-applications are defined as

$$\diamond A := \zeta^{-1}A, \qquad \overset{\zeta}{A} := A\zeta \tag{2}$$

where the juxtaposition denotes the product in $\mathcal{A}$. All the formalism to be developed hereon is motivated by definitions in (2).

The *isotope-*ζ of the algebra $\mathcal{A}$, denoted by $\mathcal{A}_\zeta$, is defined as being the underlying vector space of the algebra $\mathcal{A}$, with multiplication given by (1). The action of the isotopic algebra $\mathcal{A}_\zeta$ on physical states, generally described by elements of a Hilbert space $\mathcal{H}$—which is an ideal on which operators in $\mathcal{A}_\zeta$ acts on—comes from the definition of the isotope-ζ of an $\mathcal{A}$-module. Consider V a left unital $\mathcal{A}$-module, with respect to the composition $A\mathbf{v}$, where $A \in \mathcal{A}$, $\mathbf{v} \in V$. Here V must be a left ideal of $\mathcal{A}$. From the map

$$\begin{aligned} \mathcal{A}_\zeta \times V &\to V, \\ (A, \mathbf{v}) &\mapsto A \diamond \mathbf{v} = A\zeta^{-1}\mathbf{v}, \end{aligned} \tag{3}$$

the $\mathcal{A}$-module V becomes a left unital $\mathcal{A}_\zeta$-module V_ζ, since $\zeta \diamond \mathbf{v} = \zeta\zeta^{-1}\mathbf{v} = \mathbf{v}$, for all $\mathbf{v} \in V$.

The product

$$\begin{aligned} \diamond : \mathcal{A} \times \mathcal{A} &\to \mathcal{A}, \\ (A, B) &\mapsto A \diamond B \end{aligned} \tag{4}$$

can also be extended in order to encompass elements $\overset{\zeta}{A}, \overset{\zeta}{B} \in \mathcal{A}_\zeta$. Indeed, given $\overset{\zeta}{A}, \overset{\zeta}{B} \in \mathcal{A}_\zeta$, it is immediate that

$$\overset{\zeta}{A} \diamond \overset{\zeta}{B} = A\zeta\zeta^{-1}B\zeta = AB\zeta \in \mathcal{A}_\zeta, \tag{5}$$

i.e., with respect to the product $\diamond$, the elements $\overset{\zeta}{A}, \overset{\zeta}{B}$ inherit the structure of the product AB in $\mathcal{A}$. This concept shall be useful in order to define exterior algebras isotopic liftings in Sect. 7.

4 Non-Associative Isotopy

In this case the algebra $\mathcal{A}$ is a non-associative $\mathbb{C}$-algebra, and therefore the last equality in (1) does not hold anymore. Given $\zeta \in \mathcal{A}$ fixed, but arbitrary, the non-associative isotope-ζ of $\mathcal{A}$, denoted by $\mathcal{A}_{(\zeta)}$, is defined by the multiplication

$$A \diamond_\zeta B := A(\zeta^{-1}B) \tag{6}$$

while the ζ*-isotope* of $\mathcal{A}$, denoted by ${}_{(\zeta)}\mathcal{A}$, is defined by the relation

$$A_\zeta \diamond B := (A\zeta^{-1})B. \tag{7}$$

We verify that, while ζ is the right unit of the algebra $\mathcal{A}_{(\zeta)}$ with respect to the product given by (6), ζ is also the left unit of ${}_{(\zeta)}\mathcal{A}$ with respect to the product given by (7). The product $\diamond_\zeta$ defines uniquely the isotope-ζ $\mathcal{A}_{(\zeta)}$ of $\mathcal{A}$, while in a similar way the product ${}_\zeta\diamond$ defines the ζ-isotope ${}_{(\zeta)}\mathcal{A}$ of $\mathcal{A}$. Naturally the product $\diamond_\zeta$ can be extended to elements $\overset{\zeta}{A}, \overset{\zeta}{B}$ in the isotope-ζ $\mathcal{A}_{(\zeta)}$ of $\mathcal{A}$, in such way that for this non-associative case it follows that

$$\overset{\zeta}{A} \diamond_\zeta \overset{\zeta}{B} := \overset{\zeta}{A}(\zeta^{-1}\overset{\zeta}{B}) = (A\zeta)(\zeta^{-1}B\zeta). \tag{8}$$

In this way it is then possible to define the product $A \circ_\zeta B := (A\zeta)(\zeta^{-1}B)$ from (8), which extends the X-product introduced in the octonionic algebra $\mathbb{O}$ context, to any non-associative algebra $\mathcal{A}$. The X-product was originally introduced in order to correctly define the transformation rules for bosonic (vector) and fermionic (spinor) fields on the tangent bundle over the 7-sphere S^7 [21]. This product is also closely related to the parallel transport of sections of the tangent bundle, at $X \in S^7$, i.e., $X \in \mathbb{O}$ such that $\bar{X}X = X\bar{X} = 1$.The X-product is also shown to be twice the parallelizing torsion [23], given by the torsion tensor, and in particular, it is used to investigate the S^7 Kač–Moody algebra [21, 25] and to obtain triality maps and G_2 actions [26, 29]. Also, it leads naturally to remarkable geometric and topological properties, for instance the Hopf fibrations $S^3 \cdots S^7 \to S^4$ and $S^8 \cdots S^{15} \to S^7$ [30–32], and twistor formalism in ten dimensions [23, 24]. Generalizations of these topics are developed in [22]. We also extend the product ${}_\zeta\diamond$ to the ζ-isotope ${}_{(\zeta)}\mathcal{A}$ of $\mathcal{A}$, in such a way that for this case we have

$$\overset{\zeta}{A}_\zeta \diamond \overset{\zeta}{B} := (\overset{\zeta}{A}\zeta^{-1})\overset{\zeta}{B} = (A\zeta\zeta^{-1})(B\zeta) = A(B\zeta). \tag{9}$$

The definitions of the left unital $\mathcal{A}_{(\zeta)}$-module and the ${}_{(\zeta)}\mathcal{A}$-module for the cases given by (6, 7) follow naturally from their respective definitions.

Example 1 The octonion algebra $\mathbb{O}$ can be generated by a basis $\{\mathbf{e}_0 = 1, \mathbf{e}_a\}_{a=1}^7$ in the underlying paravector space [15, 16] $\mathbb{R} \oplus \mathbb{R}^{0,7} \hookrightarrow \mathcal{C}\ell_{0,7}$, endowed with the standard octonionic product $\circ : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$, which can be constructed using the Clifford algebras $\mathcal{C}\ell_{0,7}$ as

$$A \circ B = \langle AB(1-\psi)\rangle_{0\oplus 1}, \quad A, B \in \mathbb{R} \oplus \mathbb{R}^{0,7}, \tag{10}$$

where $\psi = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_4 + \mathbf{e}_2\mathbf{e}_3\mathbf{e}_5 + \mathbf{e}_3\mathbf{e}_4\mathbf{e}_6 + \mathbf{e}_4\mathbf{e}_5\mathbf{e}_7 + \mathbf{e}_5\mathbf{e}_6\mathbf{e}_1 + \mathbf{e}_6\mathbf{e}_7\mathbf{e}_2 + \mathbf{e}_7\mathbf{e}_1\mathbf{e}_3 \in \Lambda^3(\mathbb{R}^{0,7}) \hookrightarrow \mathcal{C}\ell_{0,7}$ and the juxtaposition denotes the Clifford product [15]. The idea of introducing the octonionic product from the Clifford product in this context is to present hereon in this example our formalism using a Clifford algebraic arena. It is now immediate to verify the usual rules between basis elements under the octonionic product:

$$\mathbf{e}_a \circ \mathbf{e}_b = \varepsilon_{ab}^{c}\mathbf{e}_c - \delta_{ab} \quad (a, b, c = 1, \dots, 7), \tag{11}$$

where we denote $\varepsilon_{ab}^{c} = 1$ for the cyclic permutations $(abc) = (124), (235), (346), (457), (561), (672)$ and (713). All the relations above can be expressed as $\mathbf{e}_a \circ \mathbf{e}_{a+1} = \mathbf{e}_{a+3 \bmod 7}$.

Now, defining $\zeta = \mathbf{e}_1$, the isotope-ζ $\mathbb{O}_{(\zeta)}$ related to the octonionic algebra $\mathbb{O}$, is given by the multiplication

$$A \diamond_\zeta B = A \circ (\mathbf{e}_1^{-1} \circ B), \tag{12}$$

and the ζ-isotope ${}_{(\zeta)}\mathbb{O}$ of $\mathbb{O}$ is defined by

$$A\,{}_\zeta\diamond B = (A \circ \mathbf{e}_1^{-1}) \circ B. \tag{13}$$

For the particular cases where $A = \mathbf{e}_2$ and $B = \mathbf{e}_4$ it follows that

$$\mathbf{e}_2 \diamond_\zeta \mathbf{e}_5 = \mathbf{e}_2 \circ (\mathbf{e}_1^{-1} \circ \mathbf{e}_5) = \mathbf{e}_2 \circ (-\mathbf{e}_6) = -\mathbf{e}_7 \tag{14}$$

while

$$\mathbf{e}_2\,{}_\zeta\diamond \mathbf{e}_5 = (\mathbf{e}_2 \circ \mathbf{e}_1^{-1}) \circ \mathbf{e}_5 = \mathbf{e}_4 \circ \mathbf{e}_5 = \mathbf{e}_7. \tag{15}$$

5 ζ-Fields and Isocomplex Fields

An isotopy of the unit $1 \in \mathcal{A}$ is defined to be the map $1 \mapsto \zeta = \zeta(x)$. For consistency of the formalism, the associative products between operators are led to their corresponding isotopic (associative) partners:

$$AB \mapsto A \diamond B = A\zeta^{-1}B, \quad \zeta \text{ fixed.} \tag{16}$$

As we have just seen, the element ζ is the unit with respect to the product $\diamond$, also denominated *isounit*. On the other hand ζ^{-1} is called *isotopic element*.

The field $\mathbb{C} = \mathbb{C}(a, +, \times)$ with elements $a \in \mathbb{C}$, ordinary sum $a_1 + a_2$ and multiplication $a_1 \times a_2 = a_1 a_2$ is isotopically lifted to the isofield $\overset{\zeta}{\mathbb{C}}(\mathfrak{a}, \overset{\zeta}{+}, \diamond)$, where the isocomplex numbers (heretofore denoted by Gothic characters) are given by $\mathfrak{a} := a\zeta$, the sum is expressed as $\mathfrak{a}_1 \overset{\zeta}{+} \mathfrak{a}_2 := (a_1 + a_2)\zeta$ and the isomultiplication by $\mathfrak{a}_1 \diamond \mathfrak{a}_2 = \mathfrak{a}_1\zeta^{-1}\mathfrak{a}_2 = (a_1 a_2)\zeta$. The fields $\mathbb{C}$ and $\overset{\zeta}{\mathbb{C}}$ are shown to be isomorphic [2]. Note that given an operator $A \in \mathcal{A}$, the isoproduct between isoscalars and such operator is given by $\mathfrak{a} \diamond A = a\zeta\zeta^{-1}A = aA$.

We should mention the effect that the lack of use of Santilli's isofield activates the theorems of catastrophic inconsistencies [9] in the case when non-canonical or non-unitary theory is not formulated on Santilli's isofields.

6 Clifford Isotopies via Associative ζ-Product

From this section on the algebra $\mathcal{A}$ is taken to be the Clifford algebra $\mathcal{C}\ell_{p,q}$ over the quadratic space $\mathbb{R}^{p,q}$. It is well known that the Lie algebra $\mathfrak{so}(p,q) \simeq \mathfrak{spin}(p,q)$ is isomorphic to the algebra $(\Lambda^2(\mathbb{R}^{p,q}), [\ ,\ ])$—where $[\ ,\ ])$ denotes the commutator—when Spin(n,0) $\simeq$ Spin(0,n) and Spin$_+$($n-1$,1) $\simeq$ Spin$_+$(1,$n-1$), $n > 4$ [27, 28]. Then besides the product given by (1), there is defined the isocommutator [2, 3, 10, 11, 13, 17] $[\ ,\ ]_\zeta$ defined by

$$[\psi_i, \psi_j]_\zeta := \psi_i \diamond \psi_j - \psi_j \diamond \psi_i = \mathfrak{c}_{ij}^k \diamond \psi_k \tag{17}$$

where ψ_i, ψ_j, ψ_k are the generators of the isotopic lifting of $\mathfrak{spin}(p,q) \hookrightarrow \mathcal{C}\ell_{p,q}$, and $\mathfrak{c}_{ij}^k := c_{ij}^k\zeta$ are the isostructure constants of the Lie isoalgebra $(\mathfrak{spin}(p,q), [\ ,\ ]_\zeta)$. Here, the c_{ij}^k denote the structure constants of the Lie algebra $\mathfrak{spin}(p,q)$.

The product $\diamond : \mathcal{C}\ell_{p,q} \times \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$ has a structure of the Clifford product, since given $\psi, \phi \in \mathcal{C}\ell_{p,q}$ it follows that

$$\overset{\zeta}{\psi} \diamond \overset{\zeta}{\phi} + \overset{\zeta}{\phi} \diamond \overset{\zeta}{\psi} = \psi\zeta\zeta^{-1}\phi\zeta + \phi\zeta\zeta^{-1}\psi\zeta = (\psi\phi + \phi\psi)\zeta = 2g(\psi,\phi)\zeta \equiv 2\mathfrak{g}(\overset{\zeta}{\psi}, \overset{\zeta}{\phi}), \tag{18}$$

where $\mathfrak{g}(\overset{\zeta}{\psi}, \overset{\zeta}{\phi}) \in \overset{\zeta}{\mathbb{R}}$ is defined to be the iso-metric $g(\psi,\phi)\zeta$, and the element $\zeta \in \mathcal{C}\ell_{p,q}$ acts as the unit with respect to the product $\diamond$.

Consider now that the algebra $\mathcal{C}\ell_{p,q}$ be endowed with the commutator $[\ ,\ ]$ given by $[\psi,\phi] = \psi\phi - \phi\psi$, $\forall \psi, \phi \in \mathcal{C}\ell_{p,q}$. The *Clifford isotopic algebra* $\mathcal{C}\ell_{p,q}^\zeta$ is defined as being the triple $(\mathcal{C}\ell_{p,q}, \diamond, [\ ,\ ]_\zeta)$, where the isocommutator

$$[\psi,\phi]_\zeta := \psi \diamond \phi - \phi \diamond \psi = \psi\zeta^{-1}\phi - \phi\zeta^{-1}\psi \tag{19}$$

can be thought as being the isotopic lifting of the commutator [,]. The Clifford algebra $\mathcal{C}\ell^{\zeta}_{p,q}$ inherits the structure of $\mathcal{C}\ell_{p,q}$, with the difference that the relations holding in $\mathcal{C}\ell_{p,q}$, with respect to the Clifford product $\psi\phi$ now are valid in the isotope $\mathcal{C}\ell^{\zeta}_{p,q}$, with product $\overset{\zeta}{\psi} \diamond \overset{\zeta}{\phi}$. Indeed,

$$[\overset{\zeta}{\psi}, \overset{\zeta}{\phi}]_\zeta = \overset{\zeta}{\psi} \diamond \overset{\zeta}{\phi} - \overset{\zeta}{\phi} \diamond \overset{\zeta}{\psi} = \psi\zeta\zeta^{-1}\phi\zeta - \phi\zeta\zeta^{-1}\psi\zeta = (\psi\phi - \phi\psi)\zeta = [\psi, \phi]\zeta. \tag{20}$$

6.1 Clifford Genotopies

Given fixed, but arbitrary elements $\xi, \zeta \in \mathcal{C}\ell_{p,q}$, (19) can be still generalized by the *genocommutator*:

$$[\psi, \phi]_{\zeta,\xi} := \psi\zeta^{-1}\phi - \phi\xi^{-1}\psi, \quad \psi, \phi \in \mathcal{C}\ell_{p,q}. \tag{21}$$

When $\xi = \zeta$ the genocommutator is led to the isocommutator given by (19). Applications concerning the Clifford genotopic admissible algebras (CGAA), defined as being the 4-tuple $(\mathcal{C}\ell_{p,q}, [\ ,\ ]_{\zeta,\xi}, (\cdot)\xi, (\cdot)\zeta)$, can be used to investigate irreversible systems. Here $(\cdot)\xi$ and $(\cdot)\zeta$ obviously denote right multiplication respectively by ξ and ζ.

7 Exterior Algebra Isotopy

It is well known that the exterior product and the (left) contraction can be defined in terms of the Clifford product respectively as

$$\mathbf{v} \wedge \psi = \frac{1}{2}(\mathbf{v}\psi + \hat{\psi}\mathbf{v}) \tag{22}$$

and

$$\mathbf{v} \lrcorner \psi = \frac{1}{2}(\mathbf{v}\psi - \hat{\psi}\mathbf{v}) \tag{23}$$

for all $\mathbf{v} \in \mathbb{R}^{p,q}$, $\psi \in \mathcal{C}\ell_{p,q}$. The most natural manner to define the exterior product isotopic lifting $\mathbf{v} \wedge \psi \mapsto \mathbf{v} \overset{\zeta}{\wedge} \psi$ is from Clifford algebras, via

$$\mathbf{v} \overset{\zeta}{\wedge} \psi := \frac{1}{2}(\mathbf{v} \diamond \psi + \hat{\psi} \diamond \mathbf{v}) = \frac{1}{2}(\mathbf{v}\zeta^{-1}\psi + \hat{\psi}\zeta^{-1}\mathbf{v}). \tag{24}$$

In a similar style, the isotopic (left) contraction is defined as being

$$\mathbf{v} \overset{\zeta}{\lrcorner} \psi = \frac{1}{2}(\mathbf{v}\zeta^{-1}\psi - \hat{\psi}\zeta^{-1}\mathbf{v}) \tag{25}$$

with an immediate analogue to the right contraction. Although these definitions above are correct from the formal viewpoint, we see from (18) that the product $\diamond$ (endowing $\mathcal{C}\ell^{\zeta}_{p,q}$) inherit the structure of the original Clifford product $\psi\phi$ only whether computed between elements $\overset{\zeta}{\psi}, \overset{\zeta}{\phi} \in \overset{\zeta}{\mathcal{C}\ell}_{p,q}$. Therefore with respect to the physical applications concerning the formalism developed, the following extensions are very useful:

$$\overset{\zeta}{\mathbf{v}} \overset{\zeta}{\wedge} \overset{\zeta}{\psi} = \frac{1}{2}(\overset{\zeta}{\mathbf{v}} \diamond \overset{\zeta}{\psi} + \hat{\overset{\zeta}{\psi}} \diamond \overset{\zeta}{\mathbf{v}}) = \frac{1}{2}(\mathbf{v}\psi + \hat{\psi}\mathbf{v})\zeta \tag{26}$$

and

$$\overset{\zeta}{\mathbf{v}} \overset{\zeta}{\lrcorner} \overset{\zeta}{\psi} = \frac{1}{2}(\overset{\zeta}{\mathbf{v}} \diamond \overset{\zeta}{\psi} - \overset{\hat{\zeta}}{\psi} \diamond \overset{\zeta}{\mathbf{v}}) = \frac{1}{2}(\mathbf{v}\psi - \hat{\psi}\mathbf{v})\zeta, \tag{27}$$

$\mathbf{v} \in \mathbb{R}^{p,q}$, $\psi \in \mathcal{C}\ell_{p,q}$. From (26) it follows that

$$\overset{\zeta}{\mathbf{u}} \overset{\zeta}{\wedge} \overset{\zeta}{\mathbf{v}} = \frac{1}{2}(\mathbf{u}\zeta\zeta^{-1}\mathbf{v}\zeta - \mathbf{v}\zeta\zeta^{-1}\mathbf{u}\zeta) = (\mathbf{u} \wedge \mathbf{v})\,\zeta, \quad \mathbf{u}, \mathbf{v} \in V. \tag{28}$$

In this way we see that the isotopic exterior product $\mathbf{u} \overset{\zeta}{\wedge} \mathbf{v}$ indeed induces the exterior product $\mathbf{u} \wedge \mathbf{v}$ *at the isospace*.

8 The Spacetime Algebra $\mathcal{C}\ell_{1,3}$

Consider an orthonormal basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in Minkowski spacetime $\mathbb{R}^{1,3}$, where $\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu = 2\eta_{\mu\nu} = 2\,\mathrm{diag}(1, -1, -1, -1)$. An arbitrary element $\Upsilon \in \mathcal{C}\ell_{1,3}$ is written as $\Upsilon = c + c^\mu\mathbf{e}_\nu + c^{\mu\nu}\mathbf{e}_{\mu\nu} + c^{\mu\nu\sigma}\mathbf{e}_{\mu\nu\sigma} + h\mathbf{e}_{0123}$, where $c, c^\mu, c^{\mu\nu}, h \in \mathbb{R}$. We use the notation $\mathbf{e}_{\mu\nu} = \mathbf{e}_\mu\mathbf{e}_\nu$, $\mathbf{e}_{\mu\nu\rho} = \mathbf{e}_\mu\mathbf{e}_\nu\mathbf{e}_\rho$ for $\mu \neq \nu \neq \rho$.

The 4-vector $\mathbf{e}_{0123}$ is denoted by $\mathbf{e}_5$ and satisfies $(\mathbf{e}_5)^2 = -1$, besides anticommutating with vectors: $\mathbf{e}_\mu\mathbf{e}_5 = -\mathbf{e}_5\mathbf{e}_\mu$. As $\mathcal{C}\ell_{1,3} \simeq \mathcal{M}(2, \mathbb{H})$, in order to obtain a representation of $\mathcal{C}\ell_{1,3}$ in terms of matrices with quaternionic entries, the primitive idempotent $f = \frac{1}{2}(1 + \mathbf{e}_0)$ is used. A left minimal ideal of $\mathcal{C}\ell_{1,3}$ is written as $I_{1,3} = \mathcal{C}\ell_{1,3}f$, which elements are written as

$$\Xi = (a^1 + a^2\mathbf{e}_{23} + a^3\mathbf{e}_{31} + a^4\mathbf{e}_{12})f + (a^5 + a^6\mathbf{e}_{23} + a^7\mathbf{e}_{31} + a^8\mathbf{e}_{12})\mathbf{e}_5 f, \tag{29}$$

where

$$\begin{aligned} &a^1 = c + c^0, \quad a^2 = c^{23} + c^{023}, \quad a^3 = -c^{13} - c^{013}, \quad a^4 = c^{12} + c^{012},\\ &a^5 = -c^{123} + c^{0123}, \quad a^6 = c^1 - c^{01}, \quad a^7 = c^2 - c^{02}, \quad a^8 = c^3 - c^{03}. \end{aligned} \tag{30}$$

Since the equality $\mathbf{e}_\mu = f\mathbf{e}_\mu f + f\mathbf{e}_\mu\mathbf{e}_5 f - f\mathbf{e}_5\mathbf{e}_\mu f - f\mathbf{e}_5\mathbf{e}_\mu\mathbf{e}_5 f$ clearly holds, from the representation $\mathbf{e}_\mu \in \mathcal{C}\ell_{1,3}$ in $\mathcal{M}(2, \mathbb{H})$ given by

$$\mathbf{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 0 & \mathfrak{i} \\ \mathfrak{i} & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & \mathfrak{j} \\ \mathfrak{j} & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & \mathfrak{k} \\ \mathfrak{k} & 0 \end{pmatrix}, \tag{31}$$

where $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ denote quaternionic units, the representations of the ideal generators

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e}_5 f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{32}$$

are obtained, and finally it is possible to write $\Upsilon \in \mathcal{C}\ell_{1,3}$ in $\mathcal{M}(2, \mathbb{H})$ as

$$\Upsilon = \begin{pmatrix} \begin{matrix} c + c^0 + (c^{23} + c^{023})\mathfrak{i} \\ +(-c^{13} - c^{013})\mathfrak{j} + (c^{12} + c^{012})\mathfrak{k} \end{matrix} & \begin{matrix} (-c^{123} - c^{0123}) + (c^1 + c^{01})\mathfrak{i} + \\ (c^2 + c^{02})\mathfrak{j} + (c^3 + c^{03})\mathfrak{k} \end{matrix} \\ \\ \begin{matrix} (-c^{123} + c^{0123}) + (c^1 - c^{01})\mathfrak{i} \\ +(c^2 - c^{02})\mathfrak{j} + (c^3 - c^{03})\mathfrak{k} \end{matrix} & \begin{matrix} (c - c^0) + (c^{23} - c^{023})\mathfrak{i} + \\ (-c^{13} + c^{013})\mathfrak{j} + (c^{12} - c^{012})\mathfrak{k} \end{matrix} \end{pmatrix}. \tag{33}$$

The $\text{Spin}_+(1,3)$ group associated with $\mathcal{C}\ell_{1,3}$ is given by

$$\text{Spin}_+(1,3) = \{R \in \mathcal{C}\ell_{1,3}^+ \mid R\tilde{R} = 1\}. \tag{34}$$

Now taking a basis $\{e_i\}$ of Euclidean space $\mathbb{R}^3$, the Clifford algebra $\mathcal{C}\ell_{3,0}$ over $\mathbb{R}^3$ is well known to be isomorphic to $\mathcal{M}(2,\mathbb{C})$ and that the quaternionic units can be written as $\mathfrak{i} = e_3e_2, \mathfrak{j} = e_3e_1, \mathfrak{k} = e_1e_2$. If the isomorphism $\mathcal{C}\ell_{3,0} \simeq \mathcal{M}(2,\mathbb{C})$ given by $e_i \mapsto \sigma_i$ is considered, where σ_i denotes the Pauli matrices given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

it follows that the matrix in (33) can be written as

$$\begin{pmatrix} a_1 & b_1 & d_1 & f_1 \\ -\bar{b}_1 & \bar{a}_1 & -\bar{f}_1 & \bar{d}_1 \\ a_2 & b_2 & d_2 & f_2 \\ -\bar{b}_2 & \bar{a}_2 & -\bar{f}_2 & \bar{d}_2 \end{pmatrix} \tag{35}$$

where $a_1 = c + c^0 + i(c^{12} + c^{012}), b_1 = -c^{13} - c^{013} + i((c^{23} + c^{023}), d_1 = -c^{123} + c^{0123} + i(c^3 - c^{03}), f_1 = c^2 - c^{02} + i(c^1 - c^{01}), a_2 = -c^{123} - c^{0123} + i(c^3 + c^{03}), b_2 = -c^{123} - c^{0123} - i(c^3 + c^{03}), d_2 = -c^{23} + c^{023} + i(-c^{13} + c^{013}), f_2 = c^{23} - c^{023} + i(-c^{13} + c^{013})$.

9 Isotopy $\mathcal{C}\ell_{1,3}^\zeta$ of $\mathcal{C}\ell_{1,3}$

In this case the basis $\{\mathbf{e}_\mu\}$ of $\mathbb{R}^{1,3}$ satisfies $\overset{\zeta}{\mathbf{e}}_\mu \diamond \overset{\zeta}{\mathbf{e}}_\nu + \overset{\zeta}{\mathbf{e}}_\nu \diamond \overset{\zeta}{\mathbf{e}}_\mu = 2\eta_{\mu\nu}\zeta$, and an arbitrary element of $\Upsilon_\zeta \in \mathcal{C}\ell_{1,3}^\zeta$ can be now written as

$$\Upsilon_\zeta = \mathfrak{c} + \mathfrak{c}^\mu \diamond \overset{\zeta}{\mathbf{e}}_\nu + \mathfrak{c}^{\mu\nu} \diamond \overset{\zeta}{\mathbf{e}}_\mu \diamond \overset{\zeta}{\mathbf{e}}_\nu + \mathfrak{c}^{\mu\nu\sigma} \diamond \overset{\zeta}{\mathbf{e}}_\mu \diamond \overset{\zeta}{\mathbf{e}}_\nu \diamond \overset{\zeta}{\mathbf{e}}_\sigma + \mathfrak{h} \diamond \overset{\zeta}{\mathbf{e}}_0 \diamond \overset{\zeta}{\mathbf{e}}_1 \diamond \overset{\zeta}{\mathbf{e}}_2 \diamond \overset{\zeta}{\mathbf{e}}_3,$$

where $\mathfrak{c}, \mathfrak{c}^\mu, \mathfrak{c}^{\mu\nu}, \mathfrak{h} \in \overset{\zeta}{\mathbb{R}}$.

The isomultivector $\overset{\zeta}{\mathbf{e}}_0 \diamond \overset{\zeta}{\mathbf{e}}_1 \diamond \overset{\zeta}{\mathbf{e}}_2 \diamond \overset{\zeta}{\mathbf{e}}_3$ is denoted by $\mathbf{e}_5^\zeta$ and satisfies $\mathbf{e}_5^\zeta \diamond \mathbf{e}_5^\zeta = -\zeta$. Now choosing a primitive idempotent, denoted by $f_\zeta = \frac{1}{2}(\zeta + \overset{\zeta}{\mathbf{e}}_0)$, a left minimal ideal $I_{1,3}^\zeta$ associated with the isotopic algebra $\mathcal{C}\ell_{1,3}^\zeta$ is written as $\mathcal{C}\ell_{1,3}^\zeta \diamond f_\zeta$, which elements are of the form

$$\begin{aligned}\Xi_\zeta &= (\mathfrak{a}^1 + \mathfrak{a}^2 \diamond \overset{\zeta}{\mathbf{e}}_2 \diamond \overset{\zeta}{\mathbf{e}}_3 + \mathfrak{a}^3 \overset{\zeta}{\mathbf{e}}_3 \diamond \overset{\zeta}{\mathbf{e}}_1 + \mathfrak{a}^4 \overset{\zeta}{\mathbf{e}}_1 \diamond \overset{\zeta}{\mathbf{e}}_2) \diamond f_\zeta \\ &\quad + (\mathfrak{a}^5 + \mathfrak{a}^6 \diamond \overset{\zeta}{\mathbf{e}}_2 \diamond \overset{\zeta}{\mathbf{e}}_3 + \mathfrak{a}^7 \diamond \overset{\zeta}{\mathbf{e}}_3 \diamond \overset{\zeta}{\mathbf{e}}_1 + \mathfrak{a}^8 \diamond \overset{\zeta}{\mathbf{e}}_1 \diamond \overset{\zeta}{\mathbf{e}}_2) \diamond \overset{\zeta}{\mathbf{e}}_5 \diamond f_\zeta, \quad \mathfrak{a}^m \in \overset{\zeta}{\mathbb{C}},\end{aligned} \tag{36}$$

where

$$\begin{aligned} &\mathfrak{a}^1 = \mathfrak{c} + \mathfrak{c}^0, && \mathfrak{a}^2 = \mathfrak{c}^{23} + \mathfrak{c}^{023}, \\ &\mathfrak{a}^3 = -\mathfrak{c}^{13} - \mathfrak{c}^{013}, && \mathfrak{a}^4 = \mathfrak{c}^{12} + \mathfrak{c}^{012}, \\ &\mathfrak{a}^5 = -\mathfrak{c}^{123} + \mathfrak{c}^{0123}, && \mathfrak{a}^6 = \mathfrak{c}^1 - \mathfrak{c}^{01}, \\ &\mathfrak{a}^7 = \mathfrak{c}^2 - \mathfrak{c}^{02}, && \mathfrak{a}^8 = \mathfrak{c}^3 - \mathfrak{c}^{03}. \end{aligned} \tag{37}$$

Since $\overset{\zeta}{\mathbf{e}}_\mu = f_\zeta \diamond \overset{\zeta}{\mathbf{e}}_\mu \diamond f_\zeta + f_\zeta \diamond \overset{\zeta}{\mathbf{e}}_\mu \diamond \mathbf{e}_5^\zeta \diamond f_\zeta - f_\zeta \diamond \mathbf{e}_5^\zeta \diamond \overset{\zeta}{\mathbf{e}}_\mu \diamond f_\zeta - f_\zeta \mathbf{e}_5^\zeta \diamond \overset{\zeta}{\mathbf{e}}_\mu \diamond \mathbf{e}_5^\zeta \diamond f_\zeta$ and from the representation of $\overset{\zeta}{\mathbf{e}}_\mu = \mathbf{e}_\mu \zeta \in \mathcal{C}\ell_{1,3}^\zeta$ in $\mathcal{M}(2,\mathbb{H})_\zeta$ it follows that

$$\overset{\zeta}{\mathbf{e}}_0 = \begin{pmatrix} \zeta_0 & \zeta_1 \\ -\zeta_2 & -\zeta_3 \end{pmatrix}, \qquad \overset{\zeta}{\mathbf{e}}_1 = \begin{pmatrix} \mathfrak{i}\zeta_2 & \mathfrak{i}\zeta_3 \\ \mathfrak{i}\zeta_0 & \mathfrak{i}\zeta_1 \end{pmatrix},$$
$$\overset{\zeta}{\mathbf{e}}_2 = \begin{pmatrix} \mathfrak{j}\zeta_2 & \mathfrak{j}\zeta_3 \\ \mathfrak{j}\zeta_0 & \mathfrak{j}\zeta_1 \end{pmatrix}, \qquad \overset{\zeta}{\mathbf{e}}_3 = \begin{pmatrix} \mathfrak{k}\zeta_2 & \mathfrak{k}\zeta_3 \\ \mathfrak{k}\zeta_0 & \mathfrak{k}\zeta_1 \end{pmatrix}, \tag{38}$$

and then it follows that

$$f_\zeta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{e}_5^\zeta \diamond f_\zeta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{39}$$

It follows that $\Upsilon \in \mathcal{C}\ell_{1,3}^\zeta$ is written in $\mathcal{M}(2,\mathbb{H})_\zeta$ as

$$\Upsilon_\zeta = \begin{pmatrix} a_1 & b_1 & d_1 & f_1 \\ -\bar{b}_1 & \bar{a}_1 & -\bar{f}_1 & \bar{d}_1 \\ a_2 & b_2 & d_2 & f_2 \\ -\bar{b}_2 & \bar{a}_2 & -\bar{f}_2 & \bar{d}_2 \end{pmatrix} \begin{pmatrix} \zeta_0 & \zeta_1 & \zeta_2 & \zeta_3 \\ -\bar{\zeta}_1 & \bar{\zeta}_0 & -\bar{\zeta}_3 & \bar{\zeta}_2 \\ \zeta_4 & \zeta_5 & \zeta_6 & \zeta_7 \\ -\bar{\zeta}_5 & \bar{\zeta}_4 & -\bar{\zeta}_7 & \bar{\zeta}_6 \end{pmatrix} \tag{40}$$

where the right hand side matrix is the complexification of $\zeta \in \mathcal{M}(2,\mathbb{H})$. The isotopic $\mathrm{Spin}_+^\zeta(1,3)$ group associated with $\mathcal{C}\ell_{1,3}^\zeta$ is now defined by

$$\mathrm{Spin}_+^\zeta(1,3) = \{R \in \mathcal{C}\ell_{1,3}^{\zeta+} \mid R \diamond \tilde{R} = \zeta\}. \tag{41}$$

9.1 Heterodimensional Isotopic Lifting of SU(3)

The isotopies of SU(3) and the proof of their local isomorphism to the conventional SU(3) symmetry were first proved in [13] and papers quoted therein.

From the examples above that show how to include the Lie algebra $\mathfrak{su}(3)$, associated with the Lie group SU(3), in $\mathbb{C} \otimes \mathcal{C}\ell_{1,3} = \mathcal{C}\ell_{1,3}(\mathbb{C})$, it is possible to construct the isotopic lifting $\mathfrak{su}(3)_\zeta \hookrightarrow \mathcal{C}\ell_{1,3}^\zeta$ as:

9.1.1 $\mathrm{SU}(3)_\zeta$*: Case 1*

It is well known that considering an orthonormal basis $\{\mathbf{e}^a\}$ of $\mathbb{R}^{p,q}$, the relation $\mathbf{e}^a\mathbf{e}^b = \mathbf{e}^a \wedge \mathbf{e}^b$ holds between the Clifford and the exterior product. Denoting $\overset{\zeta}{\mathbf{e}}{}^\mu$ by $\mathbf{e}_\zeta^\mu$, we define the isotopic lifting $\mathfrak{su}(3)_\zeta$ of $\mathfrak{su}(3)$, that generates the isotopic group $\mathrm{SU}(3)_\zeta$, as the isotopic lifting given by

$$\begin{aligned}
\lambda_\zeta^1 &= \frac{1}{2}(\mathbf{e}_\zeta^0 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^1 + i\mathbf{e}_\zeta^2 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^3), & \lambda_\zeta^2 &= \frac{1}{2}(\mathbf{e}_\zeta^0 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^2 - i\mathbf{e}_\zeta^1 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^3), \\
\lambda_\zeta^3 &= \frac{1}{2}(\mathbf{e}_\zeta^0 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^3 + i\mathbf{e}_\zeta^1 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^2), & \lambda_\zeta^4 &= \frac{1}{2}(\mathbf{e}_\zeta^0 + i\mathbf{e}_\zeta^0 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^1 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^2,) \\
\lambda_\zeta^5 &= \frac{1}{2}(i\mathbf{e}_\zeta^3 - \mathbf{e}_\zeta^1 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^2 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^3), & \lambda_\zeta^6 &= \frac{1}{2}(\mathbf{e}_\zeta^0 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^2 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^3 + i\mathbf{e}_\zeta^2), \\
\lambda_\zeta^7 &= \frac{i}{2}(\mathbf{e}_\zeta^1 + \mathbf{e}_\zeta^0 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^1 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^3), & \lambda_\zeta^8 &= \frac{i}{\sqrt{3}}\mathbf{e}_\zeta^5 + \frac{1}{2\sqrt{3}}\mathbf{e}_\zeta^0 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^3 - \frac{i}{2\sqrt{3}}\mathbf{e}_\zeta^1 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^2,
\end{aligned} \tag{42}$$

where $\mathbf{e}_\zeta^5 := \mathbf{e}_\zeta^0 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^1 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^2 \overset{\zeta}{\wedge} \mathbf{e}_\zeta^3$.

Example 2 For instance, let us calculate, e.g., the isocommutator $[\lambda^1_\zeta, \lambda^2_\zeta]_\zeta$. Using definitions in (42) and making use of the definitions (24) and (26) yields

$$\begin{aligned}
[\lambda^1_\zeta, \lambda^2_\zeta]_\zeta &= \left[\frac{1}{2}(\mathbf{e}^0_\zeta \overset{\zeta}{\wedge} \mathbf{e}^1_\zeta + i\mathbf{e}^2_\zeta \overset{\zeta}{\wedge} \mathbf{e}^3_\zeta), \frac{1}{2}(\mathbf{e}^0_\zeta \overset{\zeta}{\wedge} \mathbf{e}^2_\zeta - i\mathbf{e}^1_\zeta \overset{\zeta}{\wedge} \mathbf{e}^3_\zeta)\right]_\zeta \\
&= \frac{1}{8}[\mathbf{e}^0_\zeta \diamond \mathbf{e}^1_\zeta - \mathbf{e}^1_\zeta \diamond \mathbf{e}^0_\zeta + i\mathbf{e}^2_\zeta \diamond \mathbf{e}^3_\zeta - i\mathbf{e}^3_\zeta \diamond \mathbf{e}^2_\zeta, -i\mathbf{e}^1_\zeta \diamond \mathbf{e}^3_\zeta + i\mathbf{e}^3_\zeta \diamond \mathbf{e}^1_\zeta \\
&\quad + \mathbf{e}^0_\zeta \diamond \mathbf{e}^2_\zeta - \mathbf{e}^2_\zeta \diamond \mathbf{e}^0_\zeta]_\zeta. \qquad (43)
\end{aligned}$$

Using (20) it follows that

$$\begin{aligned}
[\lambda^1_\zeta, \lambda^2_\zeta]_\zeta &= \frac{1}{8}[\mathbf{e}^0\mathbf{e}^1 - \mathbf{e}^1\mathbf{e}^0 + i\mathbf{e}^2\mathbf{e}^3 - i\mathbf{e}^3\mathbf{e}^2, -i\mathbf{e}^1\mathbf{e}^3 + i\mathbf{e}^3\mathbf{e}^1 + \mathbf{e}^0\mathbf{e}^2 - \mathbf{e}^2\mathbf{e}^0]\zeta \\
&= \frac{1}{4}(\mathbf{e}^0\mathbf{e}^1 - \mathbf{e}^1\mathbf{e}^0 + i\mathbf{e}^2\mathbf{e}^3 - i\mathbf{e}^3\mathbf{e}^2)(-i\mathbf{e}^1\mathbf{e}^3 + i\mathbf{e}^3\mathbf{e}^1 + \mathbf{e}^0\mathbf{e}^2 - \mathbf{e}^2\mathbf{e}^0)\zeta \\
&= \frac{1}{4}(4\mathbf{e}^1\mathbf{e}^2 - 4\mathbf{e}^2\mathbf{e}^1 + 4i\mathbf{e}^0\mathbf{e}^3 - 4i\mathbf{e}^3\mathbf{e}^0)\zeta \\
&= \frac{1}{4}(2i\mathbf{e}^0 \wedge \mathbf{e}^3 - 2\mathbf{e}^1 \wedge \mathbf{e}^2)\zeta \\
&= i\left(\frac{1}{2}(\mathbf{e}^0 \wedge \mathbf{e}^3 + i\mathbf{e}^1 \wedge \mathbf{e}^2)\right)\zeta \\
&= i\left(\frac{1}{2}(\mathbf{e}^0_\zeta \overset{\zeta}{\wedge} \mathbf{e}^3_\zeta + i\mathbf{e}^1_\zeta \overset{\zeta}{\wedge} \mathbf{e}^2_\zeta)\right), \quad \text{from (28)} \\
&= i\lambda^3_\zeta. \qquad (44)
\end{aligned}$$

More generally the generators λ^a_ζ satisfy the properties $[\lambda^a_\zeta, \lambda^b_\zeta]_\zeta = if_{abc}\boldsymbol{\delta}^{-1/2}\lambda^c_\zeta$, where the $\mathfrak{su}(3)$ structure constants f_{abc} are given by $f_{123} = 2f_{147} = -2f_{156} = 2f_{246} = 2f_{257} = 2f_{345} = -2f_{367} = 2f_{458}/\sqrt{3} = 2f_{678}/\sqrt{3} = 1$ [14].

It is also immediate to note that the elements $\{\lambda^1_\zeta, \lambda^2_\zeta, \lambda^3_\zeta\}$ and λ^8_ζ generates the isotopic subalgebra $\mathfrak{su}(2)_\zeta \times \mathfrak{u}(1)_\zeta$.

9.1.2 $\mathrm{SU}(3)_\zeta$*: Case 2*

By isotopically lifting the Lie algebra $\mathfrak{su}(3)$ presented in [33, 34] above it follows that

$$\begin{aligned}
\xi^1_\zeta &= -\frac{i}{2}(\mathbf{e}^2_\zeta \overset{\zeta}{\wedge} \mathbf{e}^3_\zeta + i\mathbf{e}^0_\zeta \overset{\zeta}{\wedge} \mathbf{e}^2_\zeta \overset{\zeta}{\wedge} \mathbf{e}^3_\zeta), &\quad \xi^2_\zeta &= \frac{i}{2}(\mathbf{e}^1_\zeta \overset{\zeta}{\wedge} \mathbf{e}^3_\zeta + \mathbf{e}^0_\zeta \overset{\zeta}{\wedge} \mathbf{e}^1_\zeta \overset{\zeta}{\wedge} \mathbf{e}^3_\zeta), \\
\xi^3_\zeta &= \frac{i}{2}(\mathbf{e}^1_\zeta \overset{\zeta}{\wedge} \mathbf{e}^2_\zeta + \mathbf{e}^0_\zeta \overset{\zeta}{\wedge} \mathbf{e}^1_\zeta \overset{\zeta}{\wedge} \mathbf{e}^2_\zeta), &\quad \xi^4_\zeta &= \frac{1}{2}(\mathbf{e}^5_\zeta + i\mathbf{e}^0_\zeta \overset{\zeta}{\wedge} \mathbf{e}^3_\zeta), \\
\xi^5_\zeta &= \frac{1}{2}(\mathbf{e}^3_\zeta - i\mathbf{e}^1_\zeta \overset{\zeta}{\wedge} \mathbf{e}^2_\zeta \overset{\zeta}{\wedge} \mathbf{e}^3_\zeta), &\quad \xi^6_\zeta &= \frac{1}{2}(\mathbf{e}^2_\zeta + i\mathbf{e}^0_\zeta \overset{\zeta}{\wedge} \mathbf{e}^1_\zeta), \\
\xi^7_\zeta &= \frac{1}{2}(\mathbf{e}^1_\zeta - i\mathbf{e}^0_\zeta \overset{\zeta}{\wedge} \mathbf{e}^2_\zeta), &\quad \xi^8_\zeta &= \frac{i}{2\sqrt{3}}(2\mathbf{e}^0_\zeta + i\mathbf{e}^1_\zeta \overset{\zeta}{\wedge} \mathbf{e}^2_\zeta - i\mathbf{e}^0_\zeta \overset{\zeta}{\wedge} \mathbf{e}^1_\zeta \overset{\zeta}{\wedge} \mathbf{e}^2_\zeta).
\end{aligned} \qquad (45)$$

From the generators $\{\xi^a_\zeta, \lambda^a_\zeta\} \equiv \mho^a$ of the isotopic Lie algebra $\mathfrak{su}(3)_\zeta$, the isotopic Lie group $\mathrm{SU}(3)_\zeta$ is constructed via the *iso*exponenciation defined by

$$\exp(\overset{\zeta}{\theta}_a \diamond \mho^a), \quad \overset{\zeta}{\theta}_a \in \overset{\zeta}{\mathbb{C}}. \tag{46}$$

10 Isotopic Lifting SU$_\zeta(n)$ of SU(n)

It has been stated in [27, 28] that all the elements of the compact spin groups Spin(n,0) $\simeq$ Spin(0,n) are exponentials of bivectors when $n > 1$. Also, the same holds for the other spin groups only for $\mathrm{Spin}_+(n-1,1) \simeq \mathrm{Spin}_+(1,n-1)$, $n > 4$ [27, 28]. It is well-known that at least the Lie algebras of type $\mathfrak{spin}(p,q) \simeq \mathfrak{so}(p,q)$ associated with the spin groups above can be described as elements of a Clifford algebra endowed with the commutator $(\mathcal{C}\ell_{p,q}, [\ ,\ ])$ for $p+q=n$ big enough [1]. Such statement for the Lie algebra $\mathfrak{so}(p,q) \simeq (\Lambda^2(\mathbb{R}^{p,q}), [\ ,\ ])$ associated with the group $\mathrm{SO}(p,q)$ can be immediately proved. The group SU(n) can be constructed in the context of the Clifford algebra $(\mathcal{C}\ell_{2(p+q)}, [\ ,\ ])$, by taking a basis $\{\mathbf{e}_a\}_{a=1}^{2n}$ of $\mathbb{R}^{p,q}$, where $\mathbf{e}_a^2 = 1$ and $p+q=n$, and defining the 2-forms [33]

$$\begin{aligned} E^{pq} &= \mathbf{e}^p \wedge \mathbf{e}^q + \mathbf{e}^{p+n} \wedge \mathbf{e}^{q+n}, \\ F^{pq} &= \mathbf{e}^p \wedge \mathbf{e}^{q+n} - \mathbf{e}^{p+n} \wedge \mathbf{e}^q, \\ H^r &= \mathbf{e}^r \wedge \mathbf{e}^{r+n} - \mathbf{e}^{r+n+1} \wedge \mathbf{e}^{r+n+1} \end{aligned} \tag{47}$$

for $p,q = 1,\dots,n$, $p \neq q$ and $k = 1,\dots,n-1$. It follows the expressions

$$\begin{aligned} &[E^{pq}, E^{st}] = 2E^{qt}, \qquad [E^{pq}, F^{pq}] = -2H^q, \\ &[E^{pq}, E^{st}] = 0, \qquad\quad [H^q, H^p] = 0, \end{aligned} \tag{48}$$

and also

$$\begin{aligned} &[F^{pq}, F^{ps}] = 2E^{qs}, \qquad [H^p, E^{pq}] = -2F^{pq}, \\ &[F^{pq}, F^{st}] = 0, \qquad\quad [H^p, E^{qs}] = 2F^{qs}, \end{aligned} \tag{49}$$

where the last commutator is non-trivial only when $q = p+1$. Relations given by (48), (49) completely define the Lie algebra $\mathfrak{su}(n)$ associated with the group SU(n) [1, 33].

Now the generators of the isotopic group $\mathrm{SU}_\zeta(n)$ are written using the isotopic exterior product defined by (24), denoting $\mathbf{e}^m_\zeta = \overset{\zeta}{\mathbf{e}}{}^m$, as

$$\begin{aligned} E^{pq}_\zeta &= \mathbf{e}^p_\zeta \overset{\zeta}{\wedge} \mathbf{e}^q_\zeta + \mathbf{e}^{p+n}_\zeta \overset{\zeta}{\wedge} \mathbf{e}^{q+n}_\zeta, \\ F^{pq}_\zeta &= \mathbf{e}^p_\zeta \overset{\zeta}{\wedge} \mathbf{e}^{q+n}_\zeta - \mathbf{e}^{p+n}_\zeta \overset{\zeta}{\wedge} \mathbf{e}^q_\zeta, \\ H^r_\zeta &= \mathbf{e}^r_\zeta \overset{\zeta}{\wedge} \mathbf{e}^{r+n}_\zeta - \mathbf{e}^{r+n+1}_\zeta \overset{\zeta}{\wedge} \mathbf{e}^{r+n+1}_\zeta, \end{aligned} \tag{50}$$

where $p,q = 1,\dots,n$, $p \neq q$ and $r = 1,\dots,n-1$. It follows that the expressions

$$\begin{aligned} &[E^{pq}_\zeta, E^{st}_\zeta]_\zeta = 2E^{qt}_\zeta, \qquad [E^{pq}_\zeta, F^{pq}_\zeta]_\zeta = -2H^q_\zeta, \\ &[E^{pq}_\zeta, E^{st}_\zeta]_\zeta = 0, \qquad\quad [H^q_\zeta, H^p_\zeta]_\zeta = 0, \end{aligned} \tag{51}$$

and the relations

$$[F_\zeta^{pq}, F_\zeta^{ps}]_\zeta = 2E_\zeta^{qs}, \qquad [H_\zeta^p, E_\zeta^{pq}]_\zeta = -2F_\zeta^{pq},$$
$$[F_\zeta^{pq}, F_\zeta^{st}]_\zeta = 0, \qquad [H_\zeta^p, E_\zeta^{qs}]_\zeta = 2F_\zeta^{qs}, \tag{52}$$

completely define $\mathrm{SU}_\zeta(n)$.

11 Applications in Flavor SU(n) Group Symmetry

Consider a Hilbert space $\mathcal{H}$—and ideal with respect to the algebra defined by the operators acting on it—with elements $\{|\psi_i\rangle, \ldots\}$, where $\langle\psi_i|\psi_j\rangle \in \mathbb{C}$, and the normalized states are given by $\langle\psi_i|\psi_j\rangle = \delta_{ij}$. In order to formulate the isotopic quantum mechanics, denominated relativistic hadronic mechanics, (RHM) [13], consider now a Hilbert *iso*space $\mathcal{H}_\zeta$, which has operators acting on its elements satisfying the rule given by (16). Elements $|\overset{\zeta}{\psi}\rangle \in \mathcal{H}_\zeta$, and elements of the dual space $\langle\overset{\zeta}{\psi}| \in \mathcal{H}_\zeta^*$ satisfy

$$\langle\overset{\zeta}{\psi} \wr \overset{\zeta}{\phi}\rangle := \langle\overset{\zeta}{\psi}|\zeta^{-1}|\overset{\zeta}{\phi}\rangle\zeta \in \overset{\zeta}{\mathbb{C}}. \tag{53}$$

In this case the normalized states are given by $\langle\overset{\zeta}{\psi} \wr \overset{\zeta}{\phi}\rangle = \zeta \in \overset{\zeta}{\mathbb{C}}$. With these definitions, Santilli shows that Hermitean (observable) operators in the quantum mechanical formalism correspond to isohermitean states in RHM.

Hereon ζ-kets $\wr\cdot\rangle$ are introduced

$$\wr\psi\rangle := \zeta^{-1}|\psi\rangle \tag{54}$$

together with the eigenvalue isoequation given by

$$\overset{\zeta}{H} \diamond |\overset{\zeta}{\psi}\rangle = \overset{\zeta}{H}\zeta^{-1}|\overset{\zeta}{\psi}\rangle = \overset{\zeta}{E} \diamond |\overset{\zeta}{\psi}\rangle = E|\overset{\zeta}{\psi}\rangle, \tag{55}$$

where $|\overset{\zeta}{\psi}\rangle$ is an element of $\mathcal{H}_\zeta$ and H denotes an arbitrary operator acting on $\mathcal{H}_\zeta$.

11.1 Exact Flavor SU(3) Symmetry, Isomesons and Isobarions

Hereon the formalism used is implicitly the Clifford algebra $\mathcal{C}\ell_{1,3}(\mathbb{C})$, since SU(3) is described in terms of $\mathcal{C}\ell_{1,3}(\mathbb{C})$, as in Sects. (9.1). In Sects. (11.1) and (11.2) we use the most recent limits of quark masses given by [14], page 36:

$$1.5\ \mathrm{MeV} \le m_u \le 3.0\ \mathrm{MeV}, \qquad 3\ \mathrm{MeV} \le m_d \le 7\ \mathrm{MeV},$$
$$70\ \mathrm{MeV} \le m_s \le 110\ \mathrm{MeV}, \qquad 1.16\ \mathrm{GeV} \le m_c \le 1.34\ \mathrm{GeV}, \tag{56}$$
$$4.13\ \mathrm{GeV} \le m_b \le 4.27\ \mathrm{GeV}, \qquad 170.9\ \mathrm{GeV} \le m_t \le 177.5\ \mathrm{GeV}$$

in order to determine the isotopic element ζ, which is shown to be function of these masses. In this sense quark masses are responsible for the deformation of the algebraic structures involved, together with the induced deformation concerning the geometric structure associated with the formalism presented here.

In this section we briefly recall the lifting of SU(3) [18–20], in the context introduced in this paper, which main aim is to extend the method to an exact symmetry in the isotopic lifting of SU(6). We must emphasize that since $SU(3) \subset \mathcal{M}(3,\mathbb{C})$, and $\mathcal{M}(4,\mathbb{C}) \simeq \mathcal{C}\ell_{1,3}(\mathbb{C})$, the characterization of SU(3) in $\mathcal{C}\ell_{1,3}(\mathbb{C})$ can be done by considering the trivial 'block' immersion of elements of $\mathcal{M}(3,\mathbb{C})$ in $\mathcal{M}(4,\mathbb{C})$, as $A \mapsto \left(\begin{smallmatrix} A & \vec{0} \\ \vec{0}^T & 1 \end{smallmatrix}\right) \in \mathcal{M}(4,\mathbb{C})$, where $A \in \mathcal{M}(3,\mathbb{C})$, $\vec{0} = (0,0,0)^T$ and $1 \in \mathbb{R}$.

Considering a basis $\{|\psi_u\rangle, |\psi_d\rangle, |\psi_s\rangle\}$ of the carrier representation space of SU(3), in a Hilbert space $\mathcal{H}$, we now extend Santilli's idea [19] describing the flavor SU(3) symmetry among quarks u, d and s, in such a way that they consequently have the same mass, in isospace.

If we choose the representation of the isounit in $\mathcal{M}(3,\mathbb{C}) \hookrightarrow \mathcal{M}(4,\mathbb{C}) \simeq \mathcal{C}\ell_{1,3}(\mathbb{C})$, as being $\zeta = \mathrm{diag}(g_{11}, g_{22}, g_{33}, 1)$, the Gell-Mann isomatrices are simply representations of the elements introduced in (42), when the Weyl (chiral) representation of $\{\mathbf{e}_\mu\}$ is considered:

$$
\begin{aligned}
\lambda_1^\zeta &= \delta^{-1/2}\begin{pmatrix} 0 & g_{11} & 0 \\ g_{22} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, &\quad \lambda_2^\zeta &= \boldsymbol{\delta}^{-1/2}\begin{pmatrix} 0 & -ig_{11} & 0 \\ ig_{22} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_3^\zeta &= \boldsymbol{\delta}^{-1/2}\begin{pmatrix} g_{11} & 0 & 0 \\ 0 & -g_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, &\quad \lambda_4^\zeta &= \boldsymbol{\delta}^{-1/2}\begin{pmatrix} 0 & 0 & g_{11} \\ 0 & 0 & 0 \\ g_{33} & 0 & 0 \end{pmatrix}, \\
\lambda_5^\zeta &= \boldsymbol{\delta}^{-1/2}\begin{pmatrix} 0 & 0 & -ig_{11} \\ 0 & 0 & 0 \\ ig_{33} & 0 & 0 \end{pmatrix}, &\quad \lambda_6^\zeta &= \boldsymbol{\delta}^{-1/2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & g_{22} \\ 0 & g_{33} & 0 \end{pmatrix}, \\
\lambda_7^\zeta &= \boldsymbol{\delta}^{-1/2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -ig_{22} \\ 0 & ig_{33} & 0 \end{pmatrix}, &\quad \lambda_8^\zeta &= \frac{\boldsymbol{\delta}^{-1/2}}{\sqrt{3}}\begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & -2g_{33} \end{pmatrix},
\end{aligned}
\tag{57}
$$

and satisfy the properties $[\lambda_a^\zeta, \lambda_b^\zeta]_\zeta = 2if_{abc}\boldsymbol{\delta}^{-1/2}\lambda_c^\zeta$, where the $\mathfrak{su}(3)$ structure constants f_{abc} are given by $f_{123} = 2f_{147} = -2f_{156} = 2f_{246} = 2f_{257} = 2f_{345} = -2f_{367} = 2f_{458}/\sqrt{3} = 2f_{678}/\sqrt{3} = 1$ [14].

With the condition $\det\zeta = 1$, we endow the Gell-Mann isomatrices with a standard adjoint representation character [17, 19]. Such condition implies that ζ can be written as

$$
\zeta = \mathrm{diag}(\alpha^{-1}, \beta^{-1}, \alpha\beta, 1), \qquad \alpha, \beta \in \mathbb{R}. \tag{58}
$$

Using this choice the isonormalized isostates are given by

$$
|\overset{\zeta}{\psi}_u\rangle = \begin{pmatrix} \alpha^{-1/2} \\ 0 \\ 0 \end{pmatrix}, \qquad |\overset{\zeta}{\psi}_d\rangle = \begin{pmatrix} 0 \\ \beta^{-1/2} \\ 0 \end{pmatrix}, \qquad |\overset{\zeta}{\psi}_s\rangle = \begin{pmatrix} 0 \\ 0 \\ (\alpha\beta)^{1/2} \end{pmatrix} \tag{59}
$$

and satisfy the relations $\langle \overset{\zeta}{\psi}_i \wr \overset{\zeta}{\psi}_j\rangle = \delta_{ij}\zeta$.

Since the mass operator in $\mathfrak{su}(3)$ is given by

$$M = \frac{1}{3}(m_u + m_d + m_s)I + \frac{1}{2}(m_u - m_d)\lambda^3 + \frac{\sqrt{3}}{6}(m_u + m_d - 2m_s)\lambda^8 \\ = \mathrm{diag}(m_u, m_d, m_s), \tag{60}$$

the isotopic lifting $\mathrm{SU}(3)_\zeta$ of SU(3) has the mass operator given by

$$\overset{\zeta}{M} = \left(\frac{1}{3}(m_u + m_d + m_s)\zeta + \frac{1}{2}(m_u - m_d)\lambda^3_\zeta + \frac{\sqrt{3}}{6}(m_u + m_d - 2m_s)\lambda^8_\zeta\right)\zeta \\ = \mathrm{diag}(\alpha^{-1}m_u, \beta^{-1}m_d, \alpha\beta m_s). \tag{61}$$

In the simultaneous limits $\alpha \to 1$ and $\beta \to 1$, it can be verified that $\overset{\zeta}{M} \to M$. We now constrain the parameters α, β that compose the isounit ζ, imposing that in isospace quarks u, d and s have the *same* mass $\overset{\zeta}{m} = \alpha^{-1}m_\zeta = \beta^{-1}m_d = \alpha\beta m_s$. Then, and α, β are shown to be functions of quarks u, d, s masses, given explicitly by

$$\alpha = \left(\frac{m_u^2}{m_s m_d}\right)^{1/3}, \qquad \beta = \left(\frac{m_d^2}{m_s m_u}\right)^{1/3}. \tag{62}$$

Taking the masses values in (57), the most recent limits of α and β are given by

$$0.2204 \le \alpha \le 0.2638, \qquad 0,2768 \le \beta \le 0.3057. \tag{63}$$

The exactness, or a better approximation for the values of α and β relies on the precision in the determination of the masses m_u, m_d and m_s.

Although the masses m_u, m_d and m_s are imposed to be equal in isospace, in physical space the conventional values of quarks u, d and s masses are given by the eigenvalue isoequation

$$\overset{\zeta}{M} \diamond |\overset{\zeta}{\psi}\rangle = M\zeta\zeta^{-1}|\overset{\zeta}{\psi}\rangle = M|\overset{\zeta}{\psi}\rangle = \mathrm{diag}(m_u, m_d, m_s)|\overset{\zeta}{\psi}\rangle \tag{64}$$

or, equivalently, via expected values:

$$\langle\overset{\zeta}{\psi}_u \wr \overset{\zeta}{M} \wr \overset{\zeta}{\psi}_u\rangle = m_u, \qquad \langle\overset{\zeta}{\psi}_d \wr \overset{\zeta}{M} \wr \overset{\zeta}{\psi}_d\rangle = m_d, \qquad \langle\overset{\zeta}{\psi}_s \wr \overset{\zeta}{M} \wr \overset{\zeta}{\psi}_s\rangle = m_s. \tag{65}$$

The hypercharge operator Y is naturally extended to isospace as

$$\overset{\zeta}{Y} = \frac{1}{2\sqrt{3}}\overset{\zeta}{\lambda}_8 = \frac{1}{2\sqrt{3}}\,\mathrm{diag}(\alpha^{-1}, \beta^{-1}, -2(\alpha\beta)), \tag{66}$$

while the z isospin component I_3 is given by

$$\frac{1}{2}\overset{\zeta}{\lambda}_3 = \mathrm{diag}(\alpha, -\beta, 0). \tag{67}$$

Indeed, the expected eigenvalues for the operators above are

$$Y(u) = \langle\overset{\zeta}{\psi}_u \wr Y \wr \overset{\zeta}{\psi}_u\rangle = \frac{1}{6}, \qquad Y(d) = \langle\overset{\zeta}{\psi}_d \wr Y \wr \overset{\zeta}{\psi}_d\rangle = \frac{1}{6}, \\ Y(s) = \langle\overset{\zeta}{\psi}_s \wr Y \wr \overset{\zeta}{\psi}_s\rangle = -\frac{1}{3} \tag{68}$$

and

$$I_3(u) = \langle \overset{\zeta}{\psi}_u \wr I_3 \wr \overset{\zeta}{\psi}_u \rangle = \frac{1}{2}, \qquad I_3(d) = \langle \overset{\zeta}{\psi}_d \wr I_3 \wr \overset{\zeta}{\psi}_d \rangle = -\frac{1}{2}, \\ I_3(s) = \langle \overset{\zeta}{\psi}_s \wr I_3 \wr \overset{\zeta}{\psi}_s \rangle = 0. \tag{69}$$

The isotopic electric charge operator is obviously given by $Q = Y + I_3$.

Now denoting $|\psi\rangle$ a state describing any of the quarks $\{|\psi_u, \psi_d, \psi_s\}$, an isotopic lifting induces mesons, described by $|\psi\rangle \otimes |\bar{\psi}\rangle$, and barions, described by $|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle$, to have the corresponding states in isospace

$$|\overset{\zeta}{\psi}\rangle \otimes_\zeta |\overset{\zeta}{\overline{\psi}}\rangle \tag{70}$$

for the mesons

$$|\overset{\zeta}{\psi}\rangle \otimes_\zeta |\overset{\zeta}{\psi}\rangle \otimes_\zeta |\overset{\zeta}{\psi}\rangle, \tag{71}$$

for the barions. The symbol $\otimes_\zeta$ denotes the *iso*tensorial product between spinor fields in $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ by

$$(\cdot) \otimes_\zeta (\cdot) := \zeta^{-1}(\cdot) \otimes (\cdot)(\widetilde{\zeta^{-1}\mathbf{e}_0})^*. \tag{72}$$

The isomeson can be expressed as

$$\text{isomeson} = \zeta^{-1} \text{ meson } (\widetilde{\zeta^{-1}\mathbf{e}_0})^*. \tag{73}$$

11.2 Exact Flavor SU(6) Isotopic Symmetry

Up to now, there is no description of the generators of the group SU(6) in terms of elements of any of the minimal Clifford algebras $\mathcal{C}\ell_{1,6} \simeq \mathcal{C}\ell_{3,4} \simeq \mathcal{C}\ell_{5,2} \simeq \mathcal{M}(8,\mathbb{C})$. Up to our knowledge, there is not any explicit construction like in Sect. 9, where in [33] SU(3) is constructed inside the Dirac algebra $\mathcal{C}\ell_{1,3}$, and here we have extended it to the isotopic case. Although there is not such an explicit construction, it is still possible to consider the isotopic lifting of the generators of the representation of SU(6)—seen as a subgroup of $\mathcal{M}(6,\mathbb{C}) \hookrightarrow \mathcal{M}(8,\mathbb{C}) \simeq \mathcal{C}\ell_{1,6} \simeq \mathcal{C}\ell_{3,4} \simeq \mathcal{C}\ell_{5,2}$. Also, the characterization of SU(6) in $\mathcal{M}(8,\mathbb{C})$ can be accomplished if the trivial 'block' immersion of elements of SU(6)$\hookrightarrow \mathcal{M}(6,\mathbb{C})$ in $\mathcal{M}(8,\mathbb{C})$, as

$$B \mapsto \begin{pmatrix} B & \vec{0} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{0}^T & 0 & 1 \end{pmatrix} \in \mathcal{M}(8,\mathbb{C}), \tag{74}$$

where $B \in \mathrm{SU}(6) \hookrightarrow \mathcal{M}(6,\mathbb{C})$, $\vec{0} = (0,0,0,0,0,0)^T$, $0 \in \mathbb{C}$, and $1 \in \mathbb{R}$.

A particular case of (52), relating exterior algebra elements in $\mathbb{R}^{12,0}$ and the generators of $\mathfrak{su}(6)$ is considered in details in the Appendix.

Now a basis $\{|\psi_u\rangle, |\psi_d\rangle, |\psi_s\rangle, |\psi_c\rangle, |\psi_b\rangle, |\psi_t\rangle\}$ of the carrier representation space of SU(6) in a Hilbert space $\mathcal{H}$ is considered, an Santilli's idea [18–20] is extended to describe the flavor SU(6) symmetry among quarks u, d, s c, b, and t, in such a way that they consequently have the same mass, in isospace.

If we choose the representation of the isounit in SU(6)$\hookrightarrow \mathcal{M}(6,\mathbb{C}) \hookrightarrow \mathcal{M}(8,\mathbb{C}) \simeq \mathcal{C}\ell_{1,3}(\mathbb{C})$, as being $\zeta = \mathrm{diag}(g_{11}, g_{22}, g_{33}, g_{44}, g_{55}, 1)$, we can describe SU(6) generators by

$$\lambda_1^\zeta = \delta^{-1/2}\begin{pmatrix} 0 & g_{11} & 0 & 0 & 0 & 0\\ g_{22} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2^\zeta = \delta^{-1/2}\begin{pmatrix} 0 & -ig_{11} & 0 & 0 & 0 & 0\\ ig_{22} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3^\zeta = \delta^{-1/2}\begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 & 0\\ 0 & -g_{22} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4^\zeta = \delta^{-1/2}\begin{pmatrix} 0 & 0 & g_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ g_{33} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_5^\zeta = \delta^{-1/2}\begin{pmatrix} 0 & 0 & -ig_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ ig_{33} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_6^\zeta = \delta^{-1/2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & g_{22} & 0 & 0 & 0\\ 0 & g_{33} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_7^\zeta = \delta^{-1/2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -ig_{22} & 0 & 0 & 0\\ 0 & ig_{33} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_8^\zeta = \frac{\delta^{-1/2}}{\sqrt{3}}\begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 & 0\\ 0 & g_{22} & 0 & 0 & 0 & 0\\ 0 & 0 & -2g_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{75}$$

$$\lambda_9^\zeta = \delta^{-1/2}\begin{pmatrix} 0 & 0 & 0 & g_{11} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ g_{44} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_{10}^\zeta = \delta^{-1/2}\begin{pmatrix} 0 & 0 & 0 & -ig_{11} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ ig_{44} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{11}^\zeta = \delta^{-1/2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & g_{22} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & g_{44} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_{12}^\zeta = \delta^{-1/2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -ig_{22} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & ig_{44} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{13}^{\zeta}=\boldsymbol{\delta}^{-1/2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&g_{33}&0&0\\0&0&g_{44}&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\qquad \lambda_{14}^{\zeta}=\boldsymbol{\delta}^{-1/2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&-ig_{33}&0&0\\0&0&ig_{44}&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$\lambda_{15}^{\zeta}=\frac{\boldsymbol{\delta}^{-1/2}}{\sqrt{6}}\begin{pmatrix}g_{11}&0&0&0&0&0\\0&g_{22}&0&0&0&0\\0&0&g_{33}&0&0&0\\0&0&0&-3g_{44}&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$\lambda_{16}^{\zeta}=\delta^{-1/2}\begin{pmatrix}0&0&0&0&g_{11}&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\g_{55}&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\qquad \lambda_{17}^{\zeta}=\delta^{-1/2}\begin{pmatrix}0&0&0&0&-ig_{11}&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\ig_{55}&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$\lambda_{18}^{\zeta}=\boldsymbol{\delta}^{-1/2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&g_{22}&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&g_{55}&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\qquad \lambda_{19}^{\zeta}=\boldsymbol{\delta}^{-1/2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&-ig_{22}&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&ig_{55}&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$\lambda_{20}^{\zeta}=\boldsymbol{\delta}^{-1/2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&g_{33}&0\\0&0&0&0&0&0\\0&0&g_{55}&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$\lambda_{21}^{\zeta}=\boldsymbol{\delta}^{-1/2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-ig_{33}&0\\0&0&0&0&0&0\\0&0&ig_{55}&0&0&0\\0&0&0&0&0&0\end{pmatrix},\tag{76}$$

$$\lambda_{22}^{\zeta}=\boldsymbol{\delta}^{-1/2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&g_{44}&0\\0&0&0&g_{55}&0&0\\0&0&0&0&0&0\end{pmatrix},\qquad \lambda_{23}^{\zeta}=\boldsymbol{\delta}^{-1/2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-ig_{44}&0\\0&0&0&ig_{55}&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$\lambda_{24}^{\zeta} = \frac{\boldsymbol{\delta}^{-1/2}}{2\sqrt{6}} \begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & -4g_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{25}^{\zeta} = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & g_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ g_{66} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{26}^{\zeta} = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -ig_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ ig_{66} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{27}^{\zeta} = \boldsymbol{\delta}^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{66} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{28}^{\zeta} = \boldsymbol{\delta}^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -ig_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & ig_{66} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{29}^{\zeta} = \boldsymbol{\delta}^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{33} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{66} & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{30}^{\zeta} = \boldsymbol{\delta}^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -ig_{33} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ig_{66} & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{31}^{\zeta} = \boldsymbol{\delta}^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{44} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{66} & 0 & 0 \end{pmatrix},$$

$$\lambda_{32}^{\zeta} = \boldsymbol{\delta}^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -ig_{44} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ig_{66} & 0 & 0 \end{pmatrix}, \quad \lambda_{33}^{\zeta} = \boldsymbol{\delta}^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{55} \\ 0 & 0 & 0 & 0 & g_{66} & 0 \end{pmatrix},$$

$$\lambda_{34}^{\zeta} = \boldsymbol{\delta}^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -ig_{55} \\ 0 & 0 & 0 & 0 & ig_{66} & 0 \end{pmatrix},$$

$$\overset{\zeta}{\lambda}_{35} = \frac{\delta^{-1/2}}{2\sqrt{30}} \begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & -5g_{66} \end{pmatrix}. \tag{77}$$

With the condition $\det \zeta = 1$, we endow the Gell-Mann isomatrices with a standard adjoint representation character [17, 19]. Such condition implies that ζ can be written as

$$\zeta = \operatorname{diag}(\alpha^{-1}, \beta^{-1}, \omega^{-1}, \kappa^{-1}, \tau^{-1}, \alpha\beta\omega\kappa\tau, 1, 1), \quad \alpha, \beta, \omega, \kappa, \tau \in \mathbb{R}. \tag{78}$$

Using this choice the isonormalized isostates are given by

$$\begin{aligned} |\overset{\zeta}{\psi}_u\rangle &= \begin{pmatrix} \alpha^{-1/2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\overset{\zeta}{\psi}_d\rangle = \begin{pmatrix} 0 \\ \beta^{-1/2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\overset{\zeta}{\psi}_s\rangle = \begin{pmatrix} 0 \\ 0 \\ \omega^{-1/2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ |\overset{\zeta}{\psi}_c\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \kappa^{-1/2} \\ 0 \\ 0 \end{pmatrix}, \quad |\overset{\zeta}{\psi}_b\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tau^{-1} \\ 0 \end{pmatrix}, \quad |\overset{\zeta}{\psi}_t\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (\alpha\beta\omega\kappa\tau)^{1/2} \end{pmatrix}, \end{aligned} \tag{79}$$

and satisfy the relations $\langle \overset{\zeta}{\psi}_a \wr \overset{\zeta}{\psi}_b \rangle = \delta_{ab}\zeta$, where here δ_{ab} denotes the Kronecker delta.

Since the mass operator in SU(6) is given by

$$\begin{aligned} M &= \frac{1}{3}(m_u + m_d + m_s + m_c + m_b + m_t)I + \frac{107}{144}(m_u - m_d)\lambda_3 \\ &\quad - \frac{55\sqrt{3}}{144}(m_u + m_d - 2m_s)\lambda_8 - \frac{55\sqrt{6}}{144}(m_u + m_d + m_s - 3m_c)\lambda_{15} \\ &\quad - \frac{11\sqrt{6}}{24}(m_u + m_d + m_s + m_c - 4m_b)\lambda_{24} \\ &\quad - \frac{\sqrt{30}}{6}(m_u + m_d + m_s + m_c + m_b - 5m_t)\lambda_{35} \\ &= \operatorname{diag}(m_u, m_d, m_s, m_c, m_b, m_t), \end{aligned} \tag{80}$$

the isotopic lifting $SU(6)_\zeta$ of SU(6) has the mass operator given by

$$\begin{aligned} \overset{\zeta}{M} &= \frac{1}{3}(m_u + m_d + m_s + m_c + m_b + m_t)\zeta + \frac{107}{144}(m_u - m_d)\overset{\zeta}{\lambda}_3\zeta \\ &\quad - \frac{55\sqrt{3}}{144}(m_u + m_d - 2m_s)\overset{\zeta}{\lambda}_8\zeta - \frac{55\sqrt{6}}{144}(m_u + m_d + m_s - 3m_c)\overset{\zeta}{\lambda}_{15}\zeta \\ &\quad - \frac{11\sqrt{6}}{24}(m_u + m_d + m_s + m_c - 4m_b)\overset{\zeta}{\lambda}_{24}\zeta \end{aligned}$$

$$- \frac{\sqrt{30}}{6}(m_u + m_d + m_s + m_c + m_b - 5m_t)\lambda_{35}^{\zeta}\zeta$$
$$= \mathrm{diag}(\alpha^{-1}m_u, \beta^{-1}m_d, \omega^{-1}m_s, \kappa^{-1}m_c, \tau^{-1}m_b, \alpha\,\beta\,\omega\,\kappa\,\tau\, m_t). \tag{81}$$

By extending the process of the previous subsection to the SU(6) isotopic lifting describing an exact flavor symmetry, the mass operator in isospace, considering six quarks, is presented as

$$\overset{\zeta}{M} = M\zeta = \mathrm{diag}(\alpha^{-1}m_u, \beta^{-1}m_d, \omega^{-1}m_s, \kappa^{-1}m_c, \tau^{-1}m_b, \alpha\beta\omega\kappa\tau\, m_t) \tag{82}$$
$$\equiv \mathrm{diag}(\overset{\zeta}{m}, \overset{\zeta}{m}, \overset{\zeta}{m}, \overset{\zeta}{m}, \overset{\zeta}{m}, \overset{\zeta}{m}) \tag{83}$$

where in this case the isounit is given by $\zeta = \mathrm{diag}(\alpha^{-1}, \beta^{-1}, \omega^{-1}, \kappa^{-1}, \tau^{-1}, \alpha\beta\omega\kappa\tau)$. Imposing (83), we see that each term of the matrix in (82) must equal each other, i.e.,

$$\alpha^{-1}m_u = \beta^{-1}m_d = \omega^{-1}m_s = \kappa^{-1}m_c = \tau^{-1}m_b = \alpha\beta\omega\kappa\tau\, m_t \tag{84}$$

and in particular, let us isolate all the variables in terms of the variable α:

$$\beta = \alpha\frac{m_d}{m_u}, \qquad \omega = \alpha\frac{m_s}{m_u}, \qquad \kappa = \alpha\frac{m_c}{m_u}, \qquad \tau = \alpha\frac{m_b}{m_u}. \tag{85}$$

Substituting (85) in the last of (84) ($\alpha^{-1}m_u = \alpha\beta\omega\kappa\tau\, m_t$) yields

$$\alpha^{-1}m_u = \alpha^5\frac{m_d}{m_u^4}m_s m_c m_b m_t \tag{86}$$

implying that

$$\alpha = \left(\frac{m_u^5}{m_d m_s m_c m_b m_t}\right)^{1/6}. \tag{87}$$

In the same way it can be shown that

$$\beta = \left(\frac{m_d^5}{m_u m_s m_c m_b m_t}\right)^{1/6}, \qquad \omega = \left(\frac{m_s^5}{m_d m_u m_c m_b m_t}\right)^{1/6}, \tag{88}$$

$$\kappa = \left(\frac{m_c^5}{m_d m_s m_u m_b m_t}\right)^{1/6}, \qquad \tau = \left(\frac{m_b^5}{m_d m_s m_c m_u m_t}\right)^{1/6}. \tag{89}$$

Substituting the values of quarks masses [14] in (57) yields

$$5.945\times 10^{-3} \le \alpha \le 8.212\times 10^{-3},$$
$$1.189\times 10^{-2} \le \beta \le 1.920\times 10^{-2},$$
$$2.774\times 10^{-1} \le \omega \le 3.018\times 10^{-1},$$
$$3.676 \le \kappa \le 4.598,$$
$$486.938 \le \tau \le 677.379.$$

12 Concluding Remarks

This paper presents for the first time the isotopies of Clifford algebras with relevant applications in flavor symmetry of quarks. We have formulated the isotopic liftings in the context of Clifford algebras and highlighted the formal description concerning isotopies for non-associative general algebras. The structure of the isoalgebra identifies the mass matrices to a multiple of the identity operator. The formalism used is solely based on Clifford algebras, the more natural formalism which is able to introduce isotopies of the exterior algebra. The flavor hadronic symmetry of the six u, d, s, c, b, t quarks is shown to be exact if the isotopic group SU(6) is regarded. We have shown that the unit of the isotopic Clifford algebra is a function of the six quark masses. It illustrates how phenomenological data concerning quark masses can constrain the geometry of spacetime, where the limits constraining the parameters, that are entries of the representation of the isounit in the isotopic group SU(6), are based on the most recent limits imposed on quark masses. We assert that the formulation of other theories in isospace can bring a new class of solutions of open questions in theoretical physics.

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Appendix Clifford Algebra Generators of $\mathfrak{su}(6)$

In order to completely define $\mathfrak{su}_\zeta(6)$ in terms of the exterior algebra of $\overset{\zeta}{\mathcal{C}\ell}_{12,0}$, in the light of (52),

$$
\begin{aligned}
E_\zeta^{pq} &= \mathbf{e}_\zeta^p \overset{\zeta}{\wedge} \mathbf{e}_\zeta^q + \mathbf{e}_\zeta^{p+n} \overset{\zeta}{\wedge} \mathbf{e}_\zeta^{q+n},\\
F_\zeta^{pq} &= \mathbf{e}_\zeta^p \overset{\zeta}{\wedge} \mathbf{e}_\zeta^{q+n} - \mathbf{e}_\zeta^{p+n} \overset{\zeta}{\wedge} \mathbf{e}_\zeta^{q},\\
H_\zeta^{r} &= \mathbf{e}_\zeta^r \overset{\zeta}{\wedge} \mathbf{e}_\zeta^{r+n} - \mathbf{e}_\zeta^{r+n+1} \overset{\zeta}{\wedge} \mathbf{e}_\zeta^{r+n+1},
\end{aligned}
\tag{90}
$$

where $p, q = 1, \dots, 6$, $p \neq q$ and $r = 1, \dots, 5$, let us identify $\{\lambda_1^\zeta, \dots, \lambda_{15}^\zeta\} = \{E_\zeta^{pq}\}$, $\{\lambda_{16}^\zeta, \dots, \lambda_{30}^\zeta\} = \{F_\zeta^{pq}\}$, and $\{\lambda_{31}^\zeta, \dots, \lambda_{35}^\zeta\} = \{H_\zeta^1, H_\zeta^2, H_\zeta^3, H_\zeta^4, H_\zeta^5\}$. By this identification it can be verified that the set $\{\lambda_1^\zeta, \dots, \lambda_{35}^\zeta\}$, explicitly constructed in (76), (77) of generators of $\mathfrak{su}(6)$ satisfy

$$
\begin{aligned}
&[E_\zeta^{pq}, E_\zeta^{st}]_\zeta = 2E_\zeta^{qt}, \qquad &&[E_\zeta^{pq}, F_\zeta^{pq}]_\zeta = -2H_\zeta^q,\\
&[E_\zeta^{pq}, E_\zeta^{st}]_\zeta = 0, &&[H_\zeta^{q}, H_\zeta^{p}]_\zeta = 0,
\end{aligned}
\tag{91}
$$

and the relations

$$
\begin{aligned}
&[F_\zeta^{pq}, F_\zeta^{ps}]_\zeta = 2E_\zeta^{qs}, \qquad &&[H_\zeta^{p}, E_\zeta^{pq}]_\zeta = -2F_\zeta^{pq},\\
&[F_\zeta^{pq}, F_\zeta^{st}]_\zeta = 0, &&[H_\zeta^{p}, E_\zeta^{qs}]_\zeta = 2F_\zeta^{qs},
\end{aligned}
\tag{92}
$$

for $p, q = 1, \dots, 6$, $p \neq q$ and $r = 1, \dots, 5$.

It is well known that $\mathcal{C}\ell_{12,0} \simeq \mathcal{M}(32,\mathbb{H})$, and we included $SU(6) \hookrightarrow \mathcal{M}(6,\mathbb{C}) \hookrightarrow \mathcal{M}(8,\mathbb{C}) \hookrightarrow \mathcal{M}(32,\mathbb{H})$, via (74) *and* the inclusion

$$A \hookrightarrow \begin{pmatrix} A & 0_8 & 0_8 & 0_8 \\ 0_8 & 0_8 & 0_8 & 0_8 \\ 0_8 & 0_8 & 0_8 & 0_8 \\ 0_8 & 0_8 & 0_8 & 0_8 \end{pmatrix},$$

where $A \in \mathcal{M}(8,\mathbb{C})$, and $0_8 \equiv 0_{8\times 8} \in \mathcal{M}(8,\mathbb{C})$. Up to our knowledge, there is not any criterion or method to *explicitly* include $\mathfrak{su}(n)$ in any Clifford algebra $\mathcal{C}\ell_j$, where $j < 2n$.

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